Skeleton

Sunday, December 13, 2020 10:28 PM

Goal: Let A, B be bounded sets in $\mathbb{R}^3$ with non-empty interior.
Then there are decompositions $A = A_1 \sqcup \dots \sqcup A_n$ and $B = B_1 \sqcup \dots \sqcup B_n$
where each B_i is a rotation and translation of A_i.

Definition. Suppose that a group G acts on a set E.
Two subsets A, B of E are **equidecomposable** if

$$A = A_1 \sqcup \dots \sqcup A_n, \quad B = B_1 \sqcup \dots \sqcup B_n, \quad B_i = g_i A_i.$$

"Same set up to cutting & rotating"

Question 1. Is the unit circle $\mathbb{T}$ equidecomposable with $\mathbb{T} \setminus \{1\}$?

Question 2. Let D be a countable set of points on the circle $\mathbb{T}$.
Is $\mathbb{T} \setminus D$ equidecomposable to $\mathbb{T}$?

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Question 3. Let D be a countable set of points on the sphere.
Is its complement $\mathbb{S}^2 \setminus D$ always equidecomposable with the whole sphere $\mathbb{S}^2$?

Q4. Is the punctured unit ball $B \setminus 0$ in $\mathbb{R}^3$ equidecomposable with the entire ball, using orientation-preserving isometries?

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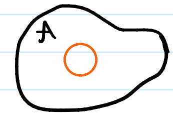
Strategy: By assumption each of A,B contains a ball & is contained in a ball.



We reduce to balls by a version of Cantor-Bernstein:

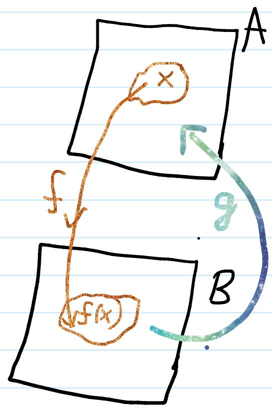
(*) If A is equidecomposable to a subset of B and vice-versa then A is equidecomposable to B.

Classical Cantor-Bernstein:

Let $A \underset{g}{\overset{f}{\rightleftarrows}} B$ be injections, then there is $X \subseteq A$ such that

$$a \mapsto \begin{cases} f(a), & a \in X \\ g^{-1}(a), & a \notin X \end{cases}$$

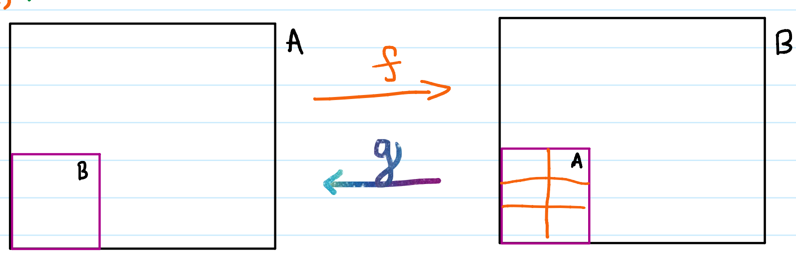
is a bijection.



Proof: Take a fixed point of the monotone function $\phi(X) = A \setminus g(Y \setminus f(X))$.

Riddle: $\phi: 2^E \circlearrowleft$
$A \subset B \rightarrow \phi(A) \subset \phi(B)$
then ϕ has a fixed point $\phi(X) = X$

Proof of (*):

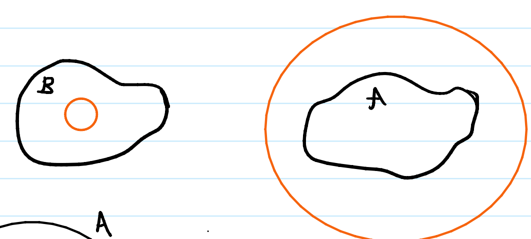


(*) If A is equidecomposable to a subset of B and vice-versa then A is equidecomposable to B.

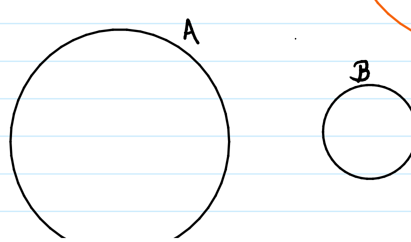
Goal: Let A,B be bounded sets in $\mathbb{R}^3$ with nonempty interior.

Then A,B are equidecomposable.

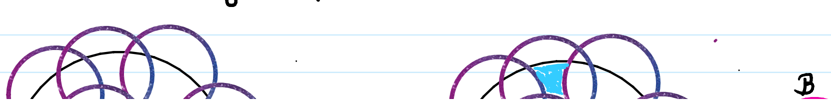
Thus we can assume that A, B are balls.

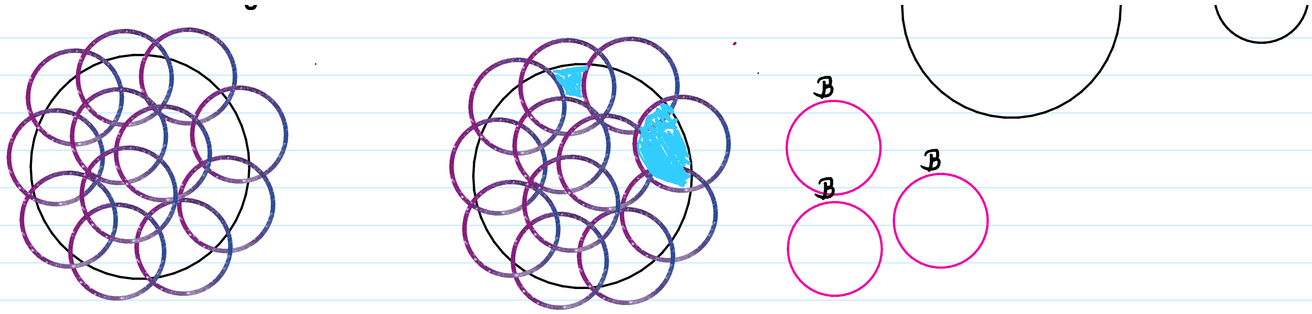


Idea. $B \leq A$ obvious, we want $A \leq B$.



Cover A by copies of B





If we find inside B many disjoint copies of B,
then we can embed each blue region in one of these copies.

Goal. Show that for every n, every ball B contains n disjoint copies of itself.
By induction, $n=2$ is enough.

Definition. $G \curvearrowright E$ paradoxical if there are disjoint $A, B \subseteq E$, both equidecomposable with E.

Riddle: Adding $A \cup B = E$ is equivalent

Goal. $\mathrm{Is}^+(\mathbb{R}^3) \curvearrowright B$ is paradoxical.

Claim. Enough to show $SO(3) \curvearrowright S^2$ is paradoxical.
Cone $\Rightarrow SO(3) \curvearrowright B \setminus \{0\}$ paradoxical and we are done.

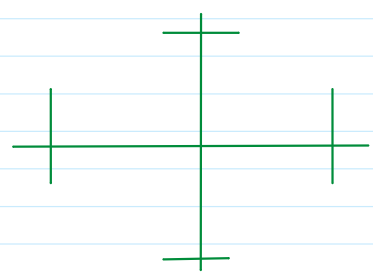
Goal. $SO(3) \curvearrowright S^2$ paradoxical.

Fact. $F_2 \leq SO(3)$.

Proposition. F_2 is paradoxical.

Proof (Animation). Cayley_backward.gif

https://upload.wikimedia.org/wikipedia/commons/0/03/Cayley_backward.gif Cayley_ba...



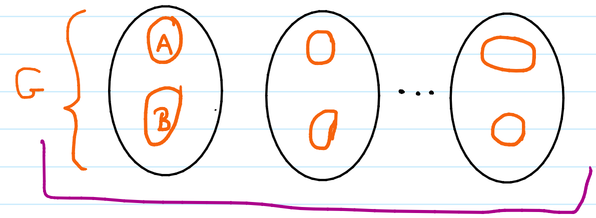
We want to transport the paradox on F_2 into a paradox of S^2.

Observation. Suppose $G \curvearrowright E$ freely. (= no nontrivial fixed points.)

If G is paradoxical, then E is paradoxical.

Proof. Decompose E into the orbits,

each orbit has a paradox.



Picture of E

$F_2 \curvearrowright \mathbb{S}^2$ is NOT free.

But it has only countably many fixed points.

F_2 is countable & A rotation has just two fixed points

F_2 acts on $\mathrm{Fixed}(F_2 \curvearrowright \mathbb{S}^2)$

$\Rightarrow$ F_2 acts on $\mathbb{S}^2 \setminus \mathrm{Fixed}(F_2 \curvearrowright \mathbb{S}^2)$ FREELY.

Thus $\mathbb{S}^2 \setminus \mathrm{Fixed}(F_2 \curvearrowright \mathbb{S}^2)$ is paradoxical.

By Question 3, $\mathbb{S}^2$ is paradoxical.